Graphs with the Erdős-Ko-Rado Property

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Abstract

For a graph G and integer $r \geq 1$ we denote the family of independent r-sets of $V(G)$ by $\mathcal{I}^{(r)}(G)$. A graph G is said to be *r-EKR* if no intersecting subfamily of $\mathcal{I}^{(r)}(G)$ is larger than the largest such family all of whose members contain some fixed $v \in V(G)$. If this inequality is always strict, then G is said to be *strictly r-EKR*. We show that if a graph G is r-EKR then its lexicographic product with any complete graph is r-EKR.

For any graph G, we define $\mu(G)$ to be the minimum size of a maximal independent vertex set. We conjecture that, if $1 \leq r \leq \frac{1}{2}\mu(G)$, then G is r-EKR, and if $r < \frac{1}{2}\mu(G)$, then G is strictly r-EKR. This is known to be true when G is an empty graph, a cycle, a path or the disjoint union of complete graphs. We show that it is also true when G is the disjoint union of a pair of complete multipartite graphs.

1 Introduction

An *independent* set in a graph $G = (V, E)$, is a subset of the vertices not containing any edges. For an integer $r \geq 1$ we denote the collection of independent r-sets of G by

$$\mathcal{I}^{(r)}(G) = \{A \subset V(G) : |A| = r \text{ and } A \text{ is an independent set}\}.$$

A subfamily $\mathcal{A}$ of $\mathcal{I}^{(r)}(G)$ is said to be *intersecting* if $A, B \in \mathcal{A}$ implies $A \cap B \neq \emptyset$. If $v \in V(G)$ then the collection of independent r-sets containing v is

$$\mathcal{I}_v^{(r)}(G) = \{A \in \mathcal{I}^{(r)}(G) : v \in A\}.$$

Such a family is called a *star*.

A graph G is said to be *r-EKR* if no intersecting family $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$ is larger than the largest star in $\mathcal{I}^{(r)}(G)$. If every intersecting family $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$ of maximum size is a star then G is said to be *strictly r-EKR*.

The classical result in this area is the Erdős-Ko-Rado theorem which can be stated as follows.

Theorem 1 (Erdős-Ko-Rado [3]) *If $G = E_n$ is the empty graph of order n, then G is r-EKR if $n \geq 2r$ and strictly r-EKR if $n > 2r$.*

There are several other recent results of this type.

Theorem 2 (Bollobás and Leader[1]) *If $n \geq r$, $t \geq 2$ and G is the disjoint union of n copies of K_t, then G is r-EKR and strictly so unless $t = 2$ and $n = r$.*

Theorem 3 (Holroyd and Talbot [4]) *If G is the disjoint union of $n \geq r$ complete graphs each of order at least two, then G is r-EKR.*

In this paper we consider the question of when a graph is r-EKR. In the next section we give the first of our two main results: if a graph G is r-EKR then its lexicographic product with any complete graph is also r-EKR.

In section 3 we present some examples showing that graphs exhibit a variety of EKR properties. These serve to motivate a conjecture we propose, giving

a lower bound on the minimum r such that a given graph G can fail to be r-EKR. This conjecture is known to be true for empty graphs, cycles, paths and disjoint unions of complete graphs. In the final section we give our second main result: this conjecture is true for disjoint unions of two complete multipartite graphs.

Throughout G is assumed to be a simple graph (without loops or multiple edges) and to have finite vertex set $V(G)$ and edge set $E(G)$. The independence number of a graph is denoted by $\alpha(G)$ and the *minimax independence number* (the minumum size of a maximal independent vertex set) by $\mu(G)$.

An *anomalous* subfamily of $\mathcal{I}^{(r)}(G)$ is an intersecting subfamily that is not a subfamily of any star. A vertex v is an *r-centre* of G if $|\mathcal{A}| \leq \left|\mathcal{I}_v^{(r)}(G)\right|$ for every intersecting subfamily $\mathcal{A}$ of $\mathcal{I}^{(r)}(G)$ and is a *strict r-centre* if $|\mathcal{A}| < \left|\mathcal{I}_v^{(r)}(G)\right|$ for every anomalous subfamily $\mathcal{A}$ of $\mathcal{I}^{(r)}(G)$.

Where no confusion is caused, we may omit the argument '(G)'.

If $\mathcal{F}$ is a family of sets then we define

$$\begin{aligned}
\mathcal{F}_x &= \{A \in \mathcal{F} \colon x \in A\}, \\
\mathcal{F}^{(r)} &= \{A \in \mathcal{F} \colon |A| = r\}, \\
\mathcal{F}_x^{(r)} &= \mathcal{F}_x \cap \mathcal{F}^{(r)}.
\end{aligned}$$

Given two graphs G and H, the *lexicographic product* $G[H]$ is constructed (informally speaking) by replacing each vertex of G with a copy of H. More formally, $V(G[H]) = V(G) \times V(H)$, where (v, w) is adjacent in $G[H]$ to (x, y) if and only if *either* v is adjacent to x in G *or* $v = x$ and w is adjacent to y in H.

It is useful to develop a generalization of this concept: rather than insisting that each vertex of G be replaced by a copy of a fixed graph, we may allow the replacement graphs to vary. For example, if we begin with G and replace each vertex $v_1, \ldots, v_k$ with a copy of a graph H and each vertex $w_1, \ldots, w_q$ with a copy of a graph J, then we denote the result by
$G[v_1, \ldots, v_k : H;\ w_1, \ldots, w_q : J]$.

2 Lexicographic products with complete graphs

We begin with a lemma concerning EKR properties of general set families, inspired by the elegant proof due to Katona [5] of the Erdős-Ko-Rado Theorem and giving it a more general context.

A family of subsets of a set S is a *q-covering* of S if each element of S belongs to exactly q sets of the family.

Lemma 4 *Let $\mathcal{F}$ be a family of r-subsets of a finite set S, let Γ be a family of subfamilies of $\mathcal{F}$, let $x \in S$, and suppose suppose that, for some q:*

(i) Γ *is a q-covering of $\mathcal{F}$;*

(ii) x *is an r-centre of each $\mathcal{G} \in \Gamma$.*

Then x is an r-centre of $\mathcal{F}$.

Proof. Let $\mathcal{A}$ be any intersecting subfamily of $\mathcal{F}$. Since Γ is a q-covering of $\mathcal{F}$, it is a q-covering of $\mathcal{A}$ and so

$$q\,|\mathcal{A}| = \sum_{\mathcal{G}\in\Gamma} |\mathcal{A}\cap\mathcal{G}|\,. \tag{1}$$

In particular,

$$q\,|\mathcal{F}_x| = \sum_{\mathcal{G}\in\Gamma} |\mathcal{G}_x|\,. \tag{2}$$

But for any intersecting subfamily $\mathcal{A}$ of $\mathcal{F}$ and any $\mathcal{G} \in \Gamma$, the family $\mathcal{A}\cap\mathcal{G}$ is an intersecting subfamily of $\mathcal{G}$, and so (since x is an r-centre of each $\mathcal{G}$)

$$|\mathcal{A}\cap\mathcal{G}| \le |\mathcal{G}_x| \;\; (\mathcal{G}\in\Gamma). \tag{3}$$

Now, (1), (2) and (3) imply (for any intersecting subfamily $\mathcal{A}$ of $\mathcal{F}$):

$$|\mathcal{A}| \le |\mathcal{F}_x|\,,$$

and so x is an r-centre of $\mathcal{F}$. □

Remark. The 'strict' extension of Lemma 4 is false. For example, let S be the vertex set of an octahedron and let $\mathcal{F}$ be the family of 3-subsets of S

corresponding to the faces. Let Γ be the 1-covering (i.e. partition) of $\mathcal{F}$ into pairs of opposite faces. Each $\mathcal{G} \in \Gamma$ is trivially EKR, and so each $x \in S$ is a strict 3-centre of each such $\mathcal{G}$. Also, each $x \in S$ is a 3-centre of $\mathcal{F}$ with $|\mathcal{F}_x| = 4$. However, there exist anomalous subfamilies of $\mathcal{F}$ of cardinality 4, namely (for each face F) the family of faces containing at least two of the vertices of F. Thus the elements of S are not strict 3-centres of $\mathcal{F}$. □

Lemma 5 *Let v be an r-centre of a graph G and let $m \in \mathbb{Z}^+$; then each vertex (v, x) ($x \in V(K_m)$ is an r-centre of the lexicographic product $G[K_m]$.*

Proof. When $m = 1$ the statement is trivial, so assume $m > 1$.

For the purposes of this proof, it is convenient to identify the vertices (in some fixed way) with the elements of the set $[n] = 1, 2, \ldots, n$, and to identify the vertices of K_m with the elements of the cyclic group $\mathbb{Z}_m$. Let $\mathcal{F}$ be the family of functions $f\colon [n] \to Z_m$. Then, for each $X \in \mathcal{I}^{(r)}(G)$ and each $f \in \mathcal{F}$, we define

$$X \circ f = \{(v, f(v)) \colon\ v \in X\}.$$

We now define an equivalence relation $\sim$ on $\mathcal{F}$ by

$$f \sim g \text{ whenever } g = g + z \text{ for some } z \in Z_m;$$

that is, $f(v) = g(v) + z$ ($v \in [n]$). We denote by Ψ the family of equivalence classes, and for each $\psi \in \Psi$ we let $\mathcal{J}_\psi$ denote the following subfamily of $\mathcal{I}^{(r)}(G[K_m])$:

$$\mathcal{J}_\psi = \{X \circ f :\ X \in \mathcal{I}^{(r)}(G),\ f \in \psi\}.$$

Each $Y \in \mathcal{I}^{(r)}(G[K_m])$ is of the form $X \circ f$ for exactly one $X \in \mathcal{I}^{(r)}(G)$ and exactly m^{n-r} functions f (each in a distinct equivalence class). That is, the family $\{\mathcal{J}_\psi :\ \psi \in \Psi\}$ is a q-covering of $\mathcal{I}^{(r)}(G[K_m])$ where $q = m^{n-r}$. By Lemma 4, it remains to show that each (v, x) ($x \in \mathbb{Z}_m$) is an r-centre of $\mathcal{J}_\psi$ for each $\psi \in \Psi$.

Let $\psi \in \Psi$ and let $\mathcal{A}$ be an intersecting subfamily of $\mathcal{J}_\psi$. Let

$$\mathcal{B} = \{X \in \mathcal{I}^{(r)}(G) :\ X \circ f \in \mathcal{A} \text{ for some } f \in \psi\}.$$

Then $\mathcal{B}$ is an intersecting subfamily of $\mathcal{I}^{(r)}(G)$, and so $|\mathcal{B}| \leq \left|\mathcal{I}_v^{(r)}(G)\right|$. If $X \in \mathcal{I}^{(r)}(G)$ and f, g are distinct elements of ψ, then $(X \circ f) \cap (X \circ g) = \emptyset$.

But $\mathcal{A}$ is intersecting; thus any two distinct elements of $\mathcal{A}$ correspond to distinct elements of $\mathcal{B}$. Hence $|\mathcal{A}| = |\mathcal{B}|$, and so

$$|\mathcal{A}| \leq \left|\mathcal{I}_v^{(r)}(G)\right|. \tag{4}$$

Let $x \in \mathbb{Z}_m$ and consider the vertex (v, x) of $G[K_m]$. For each $\psi \in \Psi$ and each $X \in \mathcal{I}^{(r)}(G)$, we have $(v, x) \in X \circ f$ for some $f \in \psi$ if and only if $X \in \mathcal{I}_v^{(r)}(G)$, in which case there is exactly one $f \in \psi$ with this property. Thus $\left|(\mathcal{J}_\psi)_{(v,x)}\right| = \left|\mathcal{I}_v^{(r)}(G)\right|$, and it follows from (4) that (v, x) is an r-centre of $\mathcal{J}_\psi$. □

Theorem 6 *If G is r-EKR and $m \geq 1$ then $G[K_m]$ is r-EKR.*

Proof. This follows directly from Lemma 5. □

It is natural to ask whether Theorem 6 extends to lexicographic products that involve replacing the vertices of G with complete graphs of variable rather than constant order. We now show that this is not unconditionally true.

Example 7 Let G be the graph with vertex set $\{v_1, \ldots, v_{13}\}$ depicted below (Figure 7). It may straightforwardly be verified that G is 3-EKR, the vertices $v_1, \ldots, v_6$ being 3-centres, with $\left|\mathcal{I}_v^{(3)}(G)\right| = 17 \quad (v = v_1, \ldots, v_6)$. The family of independent vertex 3-sets containing at least two of the vertices v_1, v_2, v_3 is of cardinality 16 and is one of two anomalous families of maximum cardinality.

Now let $m \in \mathbb{Z}^+$ and consider the graph $G[v_{13} \colon K_m]$.

Then, $\left|\mathcal{I}_v^{(3)}(G[v_{13} \colon K_m])\right| = 15 + 2m$ $(v = v_1, \ldots, v_6)$, the values for the remaining vertices being independent of m. However, the anomalous family consisting of all independent 3-sets of $G[v_{13} \colon K_m]$ containing at least two of the vertices v_1, v_2, v_3 is of cardinality $13 + 3m$. Thus, for $m > 2$, the vertices $v_1, \ldots, v_6$ of $G[v_{13} \colon K_m]$ are not 3-centres (and $G[v_{13} \colon K_m]$ is not 3-EKR).

3 Examples of EKR behaviour and a conjecture

Trivially, any graph is 1-EKR. The question of when a (non-complete) graph is 2-EKR is easy to deal with:

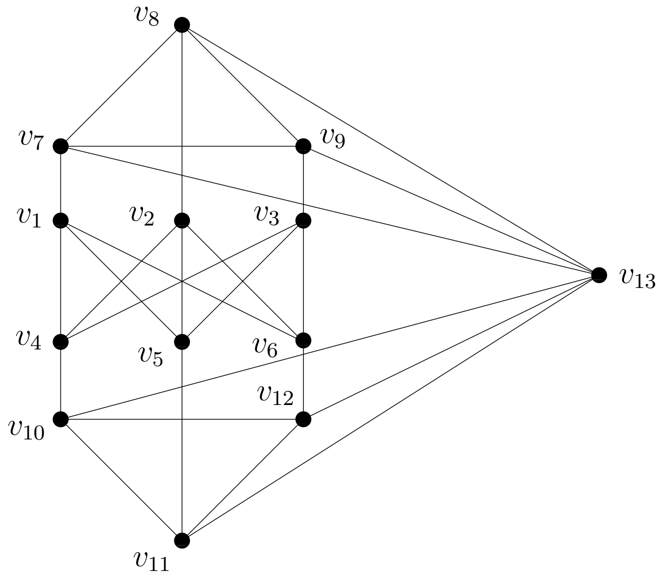

Figure 1: the graph G of Example 7

Theorem 8 *Let G be any non-complete graph of order n and with minimum degree δ.*

(i) If $\alpha = 2$, then G is strictly 2-EKR.

(ii) If $\alpha \geq 3$, then G is 2-EKR if and only if $\delta \leq n-4$ and strictly so if and only if $\delta \leq n-5$, the 2-centres being the vertices of minimum degree.

Proof. Let $\mathcal{A}$ be an anomalous family of independent vertex 2-sets. Then $|\mathcal{A}| \geq 3$, and $\mathcal{A}$ must contain the three 2-subsets of some independent 3-set; but then no other 2-set can intersect all three of these, and so $\mathcal{A}$ must consist exactly of the three 2-subsets of an independent 3-set. Thus:

(i) If $\alpha = 2$, then there is no anomalous family of independent vertex 2-sets, so G is strictly 2-EKR;

(ii) Otherwise, the anomalous families of independent vertex 2-sets are all of cardinality 3 and the result follows from the fact that, for any vertex v,

$$\left|\mathcal{I}_v^{(2)}\right| = n - 1 - d(v).$$

□

For the remainder of the paper, then, we concentrate on on the question: for $3 \leq r \leq \alpha(G)$, when is G r-EKR?

All of the graphs studied in [4], including those arising from reinterpreting [3] and [1], are α-EKR and also $\lfloor \alpha/2 \rfloor$-EKR, giving rise to the question: is this always true? The answer is no, as the following examples show.

Example 9 Let G be the graph of the regular dodecahedron (that is, the graph whose vertices and edges are those of the dodecahedron).

Then $\alpha = 8$, where $\mathcal{I}^8$ consists of the vertex sets of the five inscribed cubes of the dodecahedron. Any pair of these sets intersects on two (opposite) vertices, but any given vertex belongs to just two of them. Thus $\mathcal{I}^8$ is an anomalous family and G is not 8-EKR. We note, without proof, that if G is the graph of any of the Platonic solids other than the dodecahedron, then G α-EKR.

Example 10 Let F be the graph with vertices $v_1, \ldots, v_7$ where $v_1, \ldots, v_4$ are pairwise adjacent and v_{i+4} is adjacent only to v_i $(i = 1, 2, 3)$. (See Figure 2.)

Now let $G = F[v_1, v_2, v_3 : K_3;\ v_4 : E_4]$. Then the order of G is 16 and $\alpha = 7$, $\mu = 3$. Moreover, the families $\mathcal{I}^{(r)}$ $(4 \leq r \leq 7)$ are precisely the families of r-subsets of the unique independent 7-set. Thus, G is 7-EKR in a trivial way and (by the Erdős-Ko-Rado Theorem) is not 4-, 5- or 6-EKR. More interestingly, G fails to be 3-EKR, since no vertex belongs to more than 21 independent 3-sets but there is an anomalous family consisting of the 22 independent 3-sets containing at least two of v_5, v_6, v_7. Thus it is possible for a graph to fail to be $\lfloor \alpha/2 \rfloor$-EKR and to fail to be μ-EKR.

In each graph studied so far, when G is α-EKR, it is so in a trivial way; but this is not so in general, as the next example shows.

Example 11 Let G be the graph of the regular icosahedron. Then $\alpha = 3$. It is straightforward to check that $\left|\mathcal{I}_v^{(3)}\right| = 5$ for any vertex v, and with a little care it is possible to construct an anomalous family of four independent 3-sets and to verify that no such family can be extended to a fifth member. Thus G is (strictly) 3-EKR.

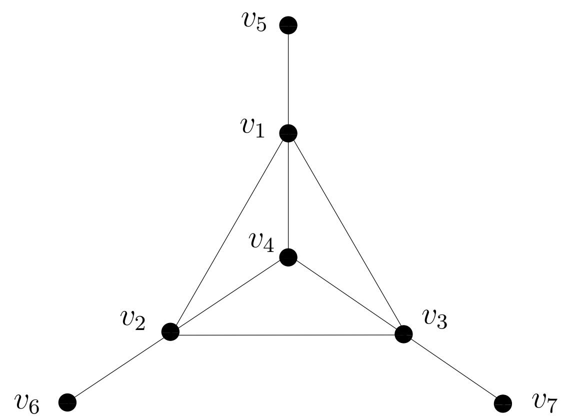

Figure 2: the graph F of Example 10

Note that the antipodal pairs of vertices of G are maximal independent sets, so that $\mu = 2$. Therefore, this example also shows that it is possible for a graph to be r-EKR for some $r > \mu$.

It is easy to vary Example 10 to produce a graph of arbitrarily large independence number that fails to be 3-EKR since, if we replace K_3 by K_p and E_4 by E_q in the generalized lexicographic construction of that example, then $\alpha = q + 3$, the maximum value of $\left|\mathcal{I}_v^{(3)}\right|$ is

$$\max\{1 + 2(p+q) + \frac{1}{2}q(q-1),\ \frac{1}{2}(q+1)(q+2)\},$$

and there is an anomalous subfamily of $\mathcal{I}^{(3)}$ of cardinality $1 + 3(p+q)$. More generally it is possible, for any $r \geq 3$, to produce a graph of arbitrarily large independence number that fails to be r-EKR. However, this does not seem to be true for the minimax independence number. We make the following conjecture.

Conjecture 12 *Let G be any graph and let $1 \leq r \leq \frac{1}{2}\mu$; then G is r-EKR (and is strictly so if $2 < r < \frac{1}{2}\mu$).*

Each of the above bounds is sharp, as our final example shows.

Example 13 Let G be the disjoint union of two copies of the complete bipartite graph $K_{3,3}$. Then (by Theorem 14 of Section 4) $\mu = 6$ and G is non-strictly 3-EKR and strictly 2-EKR, but not 4-EKR.

4 Unions of complete multipartite graphs

It seems plausible that if any graphs fail to be r-EKR, for some $r \leq \frac{1}{2}\mu$, then the smallest examples should have $\mu = \alpha$ (that is, all maximal independent vertex sets should have the same cardinality). This motivates the study of classes of graphs with this property.

Conjecture 5 is already known to hold for certain classes of graphs; in particular it holds for empty graphs and disjoint unions of complete graphs (both of which have $\mu = \alpha$). We now show that the conjecture also holds for the class of unions of pairs of complete multipartite graphs; moreover, if G is such a graph and $\mu = \alpha$, then the bound is sharp in that G fails to be r-EKR if $\frac{1}{2}\mu < r < \mu$.

Theorem 14 *Let G be a union of two complete multipartite graphs; then:*

(i) G is r-EKR if $1 \leq r \leq \frac{1}{2}\mu$;

(ii) G is strictly r-EKR if $2 < r < \frac{1}{2}\mu$;

(iii) G is not r-EKR if $\mu = \alpha$ and $\frac{1}{2}\mu < r < \mu$.

Before proving this result, we require further notation and lemmas.

Let $b_1 \geq b_2 \geq \ldots \geq b_a$. We denote by $K_a[b_1, b_2, \ldots, b_a]$ the complete a-partite graph with partite sets of sizes $b_1, b_2, \ldots, b_a$ respectively.

Let G be the disjoint union of two complete bipartite graphs, $G_1 = K_a[b_1, \ldots, b_a]$ and $G_2 = K_c[d_1, \ldots d_c]$. Denote the partite sets of G_1 by $V_1, \ldots V_a$ where $V_i = \{v_{i,1}, \ldots, v_{i,b_i}\}$ $\quad (i = 1, \ldots, a)$ and those of G_2 by $W_1, \ldots, W_c$ where $W_i = \{w_{i,1}, \ldots, w_{i,d_i}\}$ $\quad (i = 1, \ldots, c)$.

For $2 \leq i \leq a$, define $\phi_i \colon V(G) \to V(G)$ as follows.

$$\begin{aligned} \phi_i(v_{i,j}) &= v_{1,j} && (v_{i,j} \in V_i), \\ \phi_i(v) &= v && (\text{otherwise}). \end{aligned}$$

Similarly, for $2 \leq i \leq c$, define $\theta_i \colon V(G) \to V(G)$ by

$$\begin{aligned} \theta_i(w_{i,j}) &= w_{1,j} && (w_{i,j} \in W_i), \\ \theta_i(w) &= w && \text{(otherwise)}. \end{aligned}$$

With slight abuse of notation, if $A \in \mathcal{I}(G)$, we may write $\phi_i(A) = \{\phi_i(x) \colon x \in A\}$ and $\theta_i(A) = \{\theta_i(x) \colon x \in A\}$. Note that $\phi_i(A), \theta_i(A) \in \mathcal{I}(G)$. We now define the *compressions* Φ_i, Θ_i on subfamilies of $\mathcal{I}(G)$ as follows. Let $\mathcal{A} \subseteq \mathcal{I}(G)$ and let $2 \leq i \leq a$. Then

$$\Phi_i(\mathcal{A}) = \{\theta_i(A) : A \in \mathcal{A}\} \cup \{A : A, \theta_i(A) \in \mathcal{A}\}.$$

More informally, for each $A \in \mathcal{A}$ that intersects V_i, we replace A by $\phi_i(A)$ provided that $\phi_i(A)$ is not already in $\mathcal{A}$; otherwise, we leave A alone.

The compressions $\Theta_i \quad (2 \leq i \leq c)$ are similarly defined.

We now note that, if $\mathcal{A}$ is a non-empty intersecting subfamily of $\mathcal{I}^(G)$, then there is some partite set of G_1 or G_2 that intersects every set of $\mathcal{A}$; for any $A \in \mathcal{A}$ is a subset of $V_i \cap W_j$ for some i, j and now there cannot be $B, C \in \mathcal{A}$ with $B \cap V_i = \emptyset$ and $C \cap W_j = \emptyset$. By exchanging G_1 and G_2 if necessary, we may assume that some fixed V_i intersects each set of $\mathcal{A}$. Clearly, $\mathcal{B} = \Phi_i(\mathcal{A})$ is an intersecting family with $|\mathcal{B}| = |\mathcal{A}|$ such that V_1 intersects each set of $\mathcal{B}$. Thus, in investigating the sizes of intersecting subfamilies $\mathcal{A}$ of $\mathcal{I}^{(r)}(G)$, we may assume that V_1 intersects each $A \in \mathcal{A}$; such a family is said to be *standardized.*

Our first lemma says that any compression of a standardized intersecting family in $\mathcal{I}^{(r)}(G)$ is a standardized intersecting family of the same size.

Lemma 15 *Let $2 \leq i \leq c$. With the above notation, if $\mathcal{A} \subseteq \mathcal{I}^(G)$ is standardized and intersecting then so is $\Theta_i(\mathcal{A})$, and $|\Theta_i(\mathcal{A})| = |\mathcal{A}|$.*

Proof. It follows immediately from the definitions that $\Theta_i(\mathcal{A})$ is standardized and that $|\Theta_i(\mathcal{A})| = |\mathcal{A}|$. We now show that $\Theta_i(\mathcal{A})$ is intersecting.

Let $A, B \in \Theta_i(\mathcal{A})$. If $A, B \in \mathcal{A}$ then $A \cap B \neq \emptyset$. Also if $A = \theta_i(C)$ and $B = \theta_i(D)$, with $C, D \in \mathcal{A}$ and $A, B \notin \mathcal{A}$, then $C \cap D \neq \emptyset$, implying that $A \cap B \neq \emptyset$. So we may suppose that $A \in \mathcal{A} \cap \Theta_i(\mathcal{A})$ and $B \in \Theta_i(\mathcal{A}) \backslash \mathcal{A}$.

$A \in \mathcal{A} \cap \Theta_i(\mathcal{A})$ implies that $C = \theta_i(A) \in \mathcal{A}$. Also $B \in \Theta_i(\mathcal{A}) \backslash \mathcal{A}$ implies that there exists $D \in \mathcal{A}$ such that $B = \theta_i(D)$. Now if $A \cap D \subseteq W_i$ then

$C \cap D = \emptyset$, a contradiction, since $C, D \in \mathcal{A}$. So there exists $x \in (A \cap D) \backslash W_i$. But then $x \in A \cap B$ as required. Hence $\Theta_i(\mathcal{A})$ is intersecting. $\square$

A family $\mathcal{B} \subseteq \mathcal{I}(G)$ is *compressed* if $\mathcal{B}$ is fixed under every compression.

Lemma 16 *Let G be as above. If $\mathcal{A} \subseteq \mathcal{I}(G)$ is a standardized intersecting family, then there is a standardized compressed intersecting family $\mathcal{B} \subseteq \mathcal{I}(G)$ such that $|\mathcal{A}| = |\mathcal{B}|$ and $A \cap B \cap (V_1 \cup W_1) \neq \emptyset \quad (A, B \in \mathcal{B})$.*

Proof. Let $\mathcal{B} = \Theta_2 \circ \Theta_3 \circ \ldots \circ \Theta_c(\mathcal{A})$. Then, for any $A \in \mathcal{B}$ such that $A \subseteq V_1 \cup W_i$ and $i > 1$, we have $\theta_i(A) \in \mathcal{B}$, and so $\mathcal{B}$ is compressed. By Lemma 15, $\mathcal{B}$ is intersecting and $|\mathcal{B}| = |\mathcal{A}|$. Now let $A, B \in \mathcal{B}$. Suppose $A \cap B \subseteq W_i$ where $i > 1$. Then $A \cap \theta_i(B) = \emptyset$, giving a contradiction since $A, \theta_i(B) \in \mathcal{B}$. $\square$

Proof of Theorem 14

Proof of (i) Let $G = G_1 \cup G_2$ as above and let $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$ be an intersecting family. By using Theorem 8 or by direct consideration of small cases, we may assume $r \geq 3$. We shall show that

$$|\mathcal{A}| \leq \left|\mathcal{I}_x^{(r)}\right| \text{ for some } x \in V(G).$$

We may assume that $\mathcal{A}$ is standardized; by Lemma 16 we may also assume that $\mathcal{A}$ is compressed and that $A \cap B \cap (V_1 \cup W_1) \neq \emptyset \quad (A, B \in \mathcal{A})$.

Partition $\mathcal{A}$ as $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_c$ where $\mathcal{A}_0 = \{A \in \mathcal{A} \colon A \subseteq V_1\}$ and, for $1 \leq i \leq c$,

$$\mathcal{A}_i = \{A \in \mathcal{A} \colon A \cap W_i \neq \emptyset\}.$$

Correspondingly, let $\mathcal{J} = \mathcal{I}_x^{(r)}$ where $x = v_{1,1} \in V_1$, and partition $\mathcal{J}$ as $\mathcal{J}_0 \cup \mathcal{J}_1 \cup \ldots \cup \mathcal{J}_c$. Now,

$$\mu(G) = b_a + d_c \leq |V_1 \cup W_1| = b_1 + d_1. \tag{5}$$

Thus, by the Erdős-Ko-Rado Theorem (since $r \leq \frac{1}{2}\mu$), we have

$$|\mathcal{A}_0| + |\mathcal{A}_1| \leq \binom{b_1 + d_1 - 1}{r - 1} = |\mathcal{J}_0| + |\mathcal{J}_1|. \tag{6}$$

We now compare $|\mathcal{A}_i|$ with $|\mathcal{J}_i| \quad (2 \le i \le c)$. Since each A in $\mathcal{A}_i \cup \mathcal{J}_i$ intersects V_1 and W_i, we have

$$s_i \le |A \cap V_1| \le t \quad (A \in \mathcal{A}_i \cup \mathcal{J}_i)$$

where $s_i = \max\{1, r-d_i\}$, $t = \min\{r-1, b_1\}$.

For $2 \le i \le c$, $s_i \le j \le t$, let $\mathcal{A}_i^{(j)} = \{A \in \mathcal{A}_i \colon |A \cap V_1| = j\}$ and $\mathcal{B}_i^{(j)} = \{A \cap V_1 \colon A \in \mathcal{A}_i^{(j)}\}$.
Analogously, let $\mathcal{J}_i^{(j)} = \{A \in \mathcal{J}_i \colon |A \cap V_1| = j\}$ and $\mathcal{K}_i^{(j)} = \{A \cap V_1 \colon A \in \mathcal{J}_i^{(j)}\}$. Then, for $2 \le i \le c$:

$$|\mathcal{A}_i| \le \sum_{j=s_i}^{t} \left|\mathcal{B}_i^{(j)}\right| \binom{d_i}{r-j}, \tag{7}$$

$$|\mathcal{J}_i| = \sum_{j=s_i}^{t} \left|\mathcal{K}_i^{(j)}\right| \binom{d_i}{r-j} = \sum_{j=s_i}^{t} \binom{b_1-1}{j-1}\binom{d_i}{r-j}. \tag{8}$$

Since $\mathcal{A}$ is standardized and compressed, each $\mathcal{B}_i$ is intersecting, by Lemma 16. Thus, by (5) and the Erdős-Ko-Rado Theorem, we have for $2 \le i \le c,\ s_i \le j \le \frac{1}{2}b_1$:

$$\left|\mathcal{B}_i^{(j)}\right| \le \binom{b_1-1}{j-1}. \tag{9}$$

Thus, if $t \le \frac{1}{2}b_1$, then we may conclude from (6)–(9) that $|\mathcal{A}| \le |\mathcal{J}| = \left|\mathcal{I}_x^{(r)}\right|$.

Suppose now that $t > \frac{1}{2}b_1$. For $s_i \le j \le \lfloor \frac{1}{2}b_1 \rfloor,\ b_1 - j \le t$, we have

$$\left|\mathcal{A}_i^{(j)} \cup \mathcal{A}_i^{(b_1-j)}\right| \le \left|\mathcal{B}_i^{(j)}\right| \binom{d_i}{r-j} + \left|\mathcal{B}_i^{(b_1-j)}\right| \binom{d_i}{r-(b_1-j)}. \tag{10}$$

Moreover, by the intersecting property, no set in $\mathcal{B}_i^{(b_1-j)}$ can be the complement of a set in $\mathcal{B}_i^{(j)}$, and hence

$$\left|\mathcal{B}_i^{(j)}\right| + \left|\mathcal{B}_i^{(b_1-j)}\right| \le \binom{b_1}{j}. \tag{11}$$

Two cases arise.

Case 1 $\left|\mathcal{B}_i^{(b_1-j)}\right| \leq \binom{b_1-1}{b_1-j-1} = \binom{b_1-1}{j}$. Then,

$$\left|\mathcal{A}_i^{(j)} \cup \mathcal{A}_i^{(b_1-j)}\right| \leq \binom{b_1-1}{j-1}\binom{d_i}{r-j}+\binom{b_1-1}{j}\binom{d_i}{r-(b_1-j)} = \left|\mathcal{K}_i^{(j)} \cup \mathcal{K}_i^{(b_1-j)}\right|. \tag{12}$$

Case 2 $\left|\mathcal{B}_i^{(b_1-j)}\right| > \binom{b_1-1}{j}$.

Now, from $r \leq \frac{1}{2}(b_1+d_i)$, it is straightforward to deduce that

$$\binom{d_i}{r-(b_1-j)} \leq \binom{d_i}{r-j}. \tag{13}$$

Together with inequality (11), this implies that (12) still holds. Thus, $|\mathcal{A}_i| \leq |\mathcal{J}_i| \quad (2 \leq i \leq c)$. With equation (6), this gives the result $|\mathcal{A}| \leq \left|\mathcal{I}_x^{(r)}\right|$, as required.

Proof of (ii)

We now show that G is strictly r-EKR for $r < \frac{1}{2}\mu$.

First note that, for $|\mathcal{A}| = \left|\mathcal{I}_x^{(r)}\right|$, equality must hold in each of the inequalities (6), (7), (9), (11); moreover, for $r < \frac{1}{2}(b_1+d_i)$, the inequality (13) is strict. Thus, by the Erdős-Ko-Rado Theorem, for $|\mathcal{A}| = \left|\mathcal{I}_x^{(r)}\right|$, $\mathcal{A}_0 \cup \mathcal{A}_1 = \{A \subseteq V_1 \cup W_1\colon\ |A| = r, x_1 \in A\}$ for some $x_1 \in V_1$. By (7), (8), the Erdős-Ko-Rado Theorem now implies, for $2 \leq i \leq c$:

$$\mathcal{A}_0 \cup \mathcal{A}_i = \{A \subseteq V_1 \cup W_i\colon\ |A| = r, x_i \in A\}$$

for some $x_i \in V_1$. Clearly, the x_i must all be equal and the result follows.

Proof of (iii)

Since $\mu(G) = \alpha(G)$, there exist b, d such that $\mu = b+d$, $b_i = b \quad (1 \leq i \leq a)$ and $d_i = d \quad (1 \leq i \leq c)$. Let $\frac{1}{2}(b+d) < r < b+d$.

If $r < b$, let $U \subseteq V_1$ such that $|U| = r, x \notin U$. Now $s+(r-1) \geq 2r-d-1 > b-1$, so that U intersects every set of $\mathcal{I}_x^{(r)}$, which is therefore not a maximal intersecting family.

If $r \geq b$, then let $x \in V_1$ and consider the family $\mathcal{J} = (\mathcal{I}_x^{(r)} \backslash \{A \in \mathcal{I}^{(r)}\colon A \cap V_1 = \{x\}\}) \cup \{A \in \mathcal{I}^{(r)}\colon A \cap V_1 = V_1 \backslash \{x\}\}$. It is straightforward to check that $\mathcal{J}$ is anomalous, intersecting and larger than $\mathcal{I}_x(r)$. □

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